Part2 Go through invariance of plurigenera.

Par 1: Review O-T for embedded pypersurface
both classical and new-version
A little bit of history (time)

Part3: Relate new-version of O-T to
algebraic assumption. (do one computation to
show how we gonna use it)

Part 4: Main technical lemma:
Quote H-M

Part 5: Explain H-M through example

Theorem 4.3. ([D3]) *Let M be a weakly pseudo-convex n-dimensional manifold and let $f\colon M\to\mathbb{C}$ be a holomorphic function such that $\partial f\neq 0$ on $\{z{:}f(z)=0\}$. Consider two smooth functions φ and ϱ on M such that*

$$\sqrt{-1}\partial\bar{\partial}\varphi\geqslant 0 \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}\varphi\geqslant\frac{1}{\alpha}\sqrt{-1}\partial\bar{\partial}\varrho,$$

and such that $|f|^2_\varrho:=|f|^2e^{-\varrho}\leqslant e^{-\alpha}$, where $\alpha\geqslant 1$ is a constant. Then, given an $(n-1)$-form γ on $M_f:=\{z{:}f(z)=0\}$, there is an n-form Γ on M satisfying the following properties:

(a) $\Gamma|_{M_f}=\gamma\wedge df$;

(b) *we have*

$$\int_M\frac{|\Gamma|^2e^{-\varrho-\varphi}}{|f|^2_\varrho\log^2|f|^2_\varrho}\leqslant C_0(n)\int_{M_f}|\gamma|^2e^{-\varphi},$$

where $C_0(n)$ is a numerical constant depending only on the dimension.

Let X be a projective manifold and let $Y \subset X$ be a non-singular hypersurface. We assume that there exists a metric h_Y on the line bundle $\mathcal{O}_X(Y)$ associated with Y,

denoted by $h_Y = e^{-\varphi_Y}$ with respect to any local trivialization, satisfying the following conditions:

(i) If we denote by s the tautological section associated with Y, then

$$|s|^2 e^{-\varphi_Y} \leqslant e^{-\alpha}, \tag{19}$$

where $\alpha \geqslant 1$ is a real number.

(ii) There are two semi-positively curved Hermitian $\mathbb{Q}$-line bundles, say $(G_1, e^{-\varphi_{G_1}})$ and $(G_2, e^{-\varphi_{G_2}})$, such that

$$\varphi_Y = \varphi_{G_1} - \varphi_{G_2} \tag{20}$$

(cf. (17) above).

Let $F \to X$ be a line bundle, endowed with a metric h_F such that the following curvature requirements are satisfied:

$$\Theta_{h_F}(F) \geqslant 0 \quad \text{and} \quad \Theta_{h_F}(F) \geqslant \frac{1}{\alpha} \Theta_{h_Y}(Y). \tag{21}$$

Moreover, we assume the existence of real numbers $\delta_0 > 0$ and C such that

$$\varphi_F \leqslant \delta_0 \varphi_{G_2} + C; \tag{22}$$

that is to say, the poles of the metric which has the "wrong" sign in the decomposition (20) are part of the singularities of h_F. Since φ_{G_2} is locally bounded from above, we may always assume that $\delta_0 \leqslant 1$.

We denote by $\bar{h}_Y = e^{-\bar{\varphi}_Y}$ a non-singular metric on the line bundle corresponding to Y. We have the following result.

Theorem 4.1. *Let u be a section of the line bundle $\mathcal{O}_Y(K_Y + F|_Y)$ such that*

$$\int_Y |u|^2 e^{-\varphi_F} < \infty \tag{23}$$

and such that the hypotheses (19)–(22) are satisfied. Then there exists a section U of the line bundle $\mathcal{O}_X(K_X + Y + F)$ with $U|_Y = u \wedge ds$ such that for every $\delta \in (0, 1]$ we have

$$\int_X |U|^2 e^{-\delta\varphi_Y - (1-\delta)\bar{\varphi}_Y - \varphi_F} \leqslant C_\delta \int_Y |u|^2 e^{-\varphi_F}, \tag{24}$$

where the constant C_δ is given explicitly by

$$C_\delta = C_0(n)\delta^{-2} \Big(\max_X |s|^2 e^{-\bar{\varphi}_Y}\Big)^{1-\delta} \tag{25}$$

for some numerical constant $C_0(n)$ depending only on the dimension (but not on δ_0 or C in (22)).

Part 2: Let $u \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$

Recall: we get a sequence of g_k

$$\begin{cases} \Theta_{mK_{\mathcal{X}}} + \partial\bar\partial g_k \geq -\frac{1}{k}\omega & (\omega \text{ Kähler form}) \\ g_k|_{\mathcal{X}_0} \geq \log|u|^2 + C \end{cases}$$

Or $\sup g_k = 0$. C independent of k. .

Let $f_k = \frac{g_k}{m}$

$$\begin{cases} \Theta_{K_{\mathcal{X}}} + \partial\bar\partial f_k \geq \frac{-1}{mk}\omega \\ f_k|_{\mathcal{X}_0} \geq \log|u|^{2/m} + C \end{cases}$$

where u is a section of $H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$

Actually we prove a integral estimate

$$\left.\begin{aligned} \int_{\mathcal{X}} f_k \leq C \\ \omega + \Theta_{K_{\mathcal{X}}} + i\partial\bar\partial f_k \geq 0 \end{aligned}\right\} \Rightarrow \quad f_k \leq C.$$

Part 3. Climbing K_X. we need.

$$m(K_X+S+B) = K_X+S+\underbrace{(m-1)(K_X+S+B)+B}_{F}.$$

We want to apply new O-T.

Review what's the condition of O-T.

1) $\varphi_S = \varphi_{G_1} - \varphi_{G_2}$, 2) $\partial\bar\partial\varphi_F \geq 0$ 3) $\partial\bar\partial\varphi_F \geq \frac{1}{\alpha}\partial\bar\partial\varphi_S$

4) $|S|^2 e^{-\varphi_S} \leq e^{-\alpha}$. 5) $\varphi_F \leq \delta_0 \varphi_{G_2} + C$.

To achieve 3). if $\partial\bar\partial\varphi_S$ is part of $\partial\bar\partial\varphi_F$ ✓.

Now let's assume φ_{min} is a metric on K_X+S+B. $\partial\bar\partial\varphi_{min} \geq 0$

$$\varphi_S := \left(\varphi_{min} - \sum_{j\geq 2} v_j \log|w_j|^2\right)\frac{1}{v_1},$$

$$\varphi_F = (m-1)\varphi_{min} + \underbrace{\sum_{j=1} b_j \log|Y_j|^2}_{\varphi_B}$$

$$\varphi_{G_1} = \frac{\varphi_{min}}{v_1}, \quad \varphi_{G_2} = \frac{1}{v_1}\sum_{j\geq 2} v_j \log|w_j|^2.$$

Then we check 3); 5).

3) $\partial\bar\partial\varphi_F \geq (m-1)\varphi_{min} \geq (m-1)v_1\varphi_{G_1} \geq (m-1)v_1 \varphi_S$ $\boxed{\text{choose } \alpha \text{ small}}$

5) $\varphi_F \leq C + \sum_{j=1} b_j \log|Y_j|^2$

$\leq \delta_0 \varphi_{G_2}$. $\boxed{\text{choose } \delta_0 < \min\left(\frac{b_j}{v_j}\right) v_1}$

Finally we need $\int_S |u|^2 e^{-\varphi_F} < +\infty$

RK: If φ_{min} is Smooth. the

the singularity of φ_F is just $\sum_{j\geq 1} b_j \log|r_j|^2$.

However $(X, S+B)$ is plt or $b_j < 1$.

Then $\int_S |u|^2 e^{-\varphi_F} < +\infty$ ✓

Hence our goal is to find some "good" φ_{min}

Such that the integrability holds.

And define $\varphi_{G_1}, \varphi_{G_2}, \varphi_F, \varphi_S$ as above.

RK: Why not just define (Remember $W_1 = S$)

$\varphi_{G_1} = \varphi_S = \frac{1}{v_1} v_1 \log|w_1|^2$, $\varphi_{G_2} = 0$

$\varphi_{min} = \sum_{j\geq 1} v_i \log|W_i|^2$, $\varphi_F = (m-1)\varphi_{min} + \varphi_B$.

Then 1), 2), 3), 4), 5) still holds (Since φ_S is part of φ_{min})

The Reason is $\int_S |u|^2 e^{-\varphi_F} < +\infty$ fails.

Since we have no estimate of $\sum_{j\geq 2} v_i \log|W_i|^2 \Big|_S$

Part 4.

Main technical lemma: (Thm 5.3 of DHP)

$(X, S+B)$ plt, $K_X+S+B \sim_{\mathbb{Q}} \sum \nu^j W_j$ $\quad B=\sum b^j Y_j$

1) $S = W_1$, $\sum_{j\geq 2} W_j \subseteq B$.

2) $u \in H^0(X, m(K_X+S+B))$, m fixed

3) Fix smooth metric h on K_X+S+B,

and there exists $\{T_k\}_{k\geq 1} \subset L^1(X)$ s.t

$$\Theta_h + i\partial\bar\partial T_k \geq \frac{-\omega}{mk}$$

$$T_k|_S \geq C(k) + \log|u|^{2/m}$$

Then there exists $(f_k)_{k\geq 1} \in L^1(X)$, s.t

① $\sup_X f_k = 0$

$$\Theta_h + i\partial\bar\partial f_k \geq -\frac{\omega}{km}$$

② $f_k|_S$ is well defined, and

$$f_k|_S \geq \boxed{C} + \log|u|^{2/m}$$

independent of m.

RK: Assumption 3) is not easily achieved under the ass of PLT.

THEOREM 5.3. *Let $\{S, Y_j\}_j$ be smooth hypersurfaces of X with normal crossings. Let $0<b^j<1$ be rational numbers and h be a Hermitian metric satisfying the following properties*:

(1) *We have $K_X+S+\sum_j b^j Y_j \equiv \sum_j \nu^j W_j$, where ν^j are positive rational numbers, and $\{W_j\}_j \subset \{S, Y_j\}_j$.*

(2) *There exist a positive integer m_0 such that $m_0(K_X+S+\sum_j b^j Y_j)$ is a Cartier divisor, and a non-identically zero section u of $\mathcal{O}_S(m_0(K_S+\sum_j b^j Y_j|_S))$.*

(3) *h is a non-singular metric on the $\mathbb{Q}$-line bundle $K_X+S+\sum_j b^j Y_j$, and there exists a sequence $\{\tau_m\}_{m\geqslant 1} \subset L^1(X)$ such that*

$$\Theta_h\Big(K_X+S+\sum_j b^j Y_j\Big)+\sqrt{-1}\partial\bar\partial\tau_m \geqslant -\frac{1}{m}\omega$$

as currents on X, the restriction $\tau_m|_S$ is well defined and we have

$$\tau_m|_S \geqslant \boxed{C(m)}+\log|u|^{2/m_0}, \tag{39}$$

where $C(m)$ is a constant, which is allowed to depend on m.

In my notation, I use m for m_0

k for m

Then there exists a constant $C<0$ independent of m and a sequence of functions $\{f_m\}_{m\geqslant 1} \subset L^1(X)$ satisfying the following properties:

(i) *We have $\sup_X f_m = 0$, and moreover*

$$\Theta_h\Big(K_X+S+\sum_j b^j Y_j\Big)+\sqrt{-1}\partial\bar\partial f_m \geqslant -\frac{1}{m}\omega$$

as currents on X.

(ii) *The restriction $f_m|_S$ is well defined and we have*

$$f_m|_S \geqslant C+\log|u|^{2/m_0}. \tag{40}$$

Some explaination of assumption 3) above

$(X, S+B)$ plt, $K_X+S+B \sim_{\mathbb{Q}} \sum \nu^j w_j$

$B = b^j Y_j$.

1) $S = W_1$ $\sum_{j\geq 2} W_j \subseteq B$.

2) $u \in H^0(X, m(K_X+S+B))$

3) Fix smooth metric h on K_X+S+B, and there exists $\{T_k\}_{k\geq 1} \subset L^1(X)$ s.t

$$\Theta_h + i\partial\bar\partial T_k \geq \frac{-\omega}{mk}, \quad \omega \text{ is Kähler form.}$$

$$T_k|_S \geq C(k) + \log|u|^{2/m}$$

Ex: equivalently. (*)

$$\exists \ \hat{u}_i \in H^0(X, mk(K_X+S+B)+A)$$

$$\text{s.t } \hat{u}_i|_S = u^k \otimes s_i$$

where s_i is fixed such that $u^k \otimes s_i$ has no base point on S.

$\partial\bar\partial \varphi_A = \omega > 0$ Kähler

$s_i \in H^0(X, A)$.

H-M

Part 5:

Let $(X, \Delta = S + A + B)$ plt pair, $\lfloor \Delta \rfloor = S$, (X, S) log smooth, $A \geq 0$ general Ample $\mathbb{Q}$-div. $B \geq 0$ $\mathbb{Q}$ div, $(S, \Omega + A|_S)$ is canonical, where $\Omega = (\Delta - S)|_S$. Assume stable locus of $K_X + \Delta$ does not contain S,

obstruction divisor

Let $F := \lim F_{l!}$

where $F_m := \mathrm{Fix}(|m(K_X+\Delta)|_S)/m$

If $\varepsilon > 0$ is rational such that $K_X + \Delta + \frac{A}{\varepsilon}$ ample, and if Φ is any $\mathbb{Q}$-div on S and $k > 0$, such that

(1) $k\Delta$, $k\Phi$ cartier

(2) $\Omega \wedge \lambda F \leq \Phi \leq \Omega$, where $\lambda = 1 - \varepsilon/k$

Theorem 6.3. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety Z, where $(X, \Delta = S + A + B)$ is a purely log terminal pair, $S = \llcorner \Delta \lrcorner$ is irreducible, (X, S) is log smooth, $A \geq 0$ is a general ample $\mathbb{Q}$-divisor, $B \geq 0$ is a $\mathbb{Q}$-divisor and $(S, \Omega + A|_S)$ is canonical, where $\Omega = (\Delta - S)|_S$. Assume that the stable base locus of $K_X + \Delta$ does not contain S. Let $F = \lim F_{l!}$, where, for any positive and sufficiently divisible integer m, we let*

$$F_m = \mathrm{Fix}(|m(K_X + \Delta)|_S)/m.$$

If $\epsilon > 0$ is any rational number such that $\epsilon(K_X + \Delta) + A$ is ample and if Φ is any $\mathbb{Q}$-divisor on S and $k > 0$ is any integer such that

1. *both $k\Delta$ and $k\Phi$ are Cartier, and*
2. *$\Omega \wedge \lambda F \leq \Phi \leq \Omega$, where $\lambda = 1 - \epsilon/k$,*

then

$$|k(K_S + \Omega - \Phi)| + k\Phi \subset |k(K_X + \Delta)|_S.$$

Rk: To get $\varepsilon(K_X+\Delta)+A$ ample, usually we need ε very small
Then $\lambda \sim 1$, Then $\Omega \wedge \lambda F$ is very close to $\Omega \wedge F$. However if
we chose $\Phi = \Omega \wedge F$, (2) always holds, no matter what's ε.

We now turn to the description of the main result of this paper (cf. Theorem 1.7) which we believe is of independent interest.

Let X be a smooth variety, and let $S+B$ be a $\mathbb{Q}$-divisor with simple normal crossings, such that $S = \lfloor S+B \rfloor$,

$$K_X + S + B \in \mathrm{Psef}(X) \quad \text{and} \quad S \not\subset N_\sigma(K_X + S + B).$$

We consider a log-resolution $\pi\colon \widetilde{X} \to X$ of $(X, S+B)$, so that we have

$$K_{\widetilde{X}} + \widetilde{S} + \widetilde{B} = \pi^\star(K_X + S + B) + \widetilde{E},$$

where $\widetilde{S}$ is the proper transform of S. Moreover $\widetilde{B}$ and $\widetilde{E}$ are effective $\mathbb{Q}$-divisors, the components of $\widetilde{B}$ are disjoint and $\widetilde{E}$ is π-exceptional.

Following [HM2] and [P2], if we consider the *extension obstruction divisor*

$$\Xi := N_\sigma(\|K_{\widetilde{X}} + \widetilde{S} + \widetilde{B}\|_{\widetilde{S}}) \wedge \widetilde{B}|_{\widetilde{S}},$$

then we have the following result.

Theorem 1.7. (Extension theorem) *Let X be a smooth variety and $S+B$ be a $\mathbb{Q}$-divisor with simple normal crossings such that*

(1) *$(X, S+B)$ is plt (i.e. S is a prime divisor with $\mathrm{mult}_S(S+B) = 1$ and $\lfloor B \rfloor = 0$);*

(2) *there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} K_X + S + B$ such that*

$$S \subset \mathrm{Supp}(D) \subset \mathrm{Supp}(S+B);$$

(3) *S is not contained in the support of $N_\sigma(K_X + S + B)$ (i.e. for any ample divisor A and any rational number $\varepsilon > 0$, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} K_X + S + B + \varepsilon A$ whose support does not contain S).*

Let m be an integer such that $m(K_X + S + B)$ is a Cartier divisor, and let u be a section of $m(K_X + S + B)|_S$ such that

$$Z_{\pi^\star(u)} + m\widetilde{E}|_{\widetilde{S}} \geqslant m\Xi,$$

where we denote by $Z_{\pi^\star(u)}$ the zero divisor of the section $\pi^\star(u)$. Then u extends to a section of $m(K_X + S + B)$.

(We can easily apply HM's result to get assumption 3) of PLT.

Assumption of PLT

1) $K_X + S + B + \varepsilon A \sim_{\mathbb{Q}} M \geq 0$

$S \not\subseteq M$

2) For simplicity, assume components of B doesn't intersect with each each

Let $N_\sigma \| K_X + S + B \|_S := \lim_{m} \frac{\operatorname{Fix} |m(K_X + S + B)|_S}{m}$

Obstruction div:

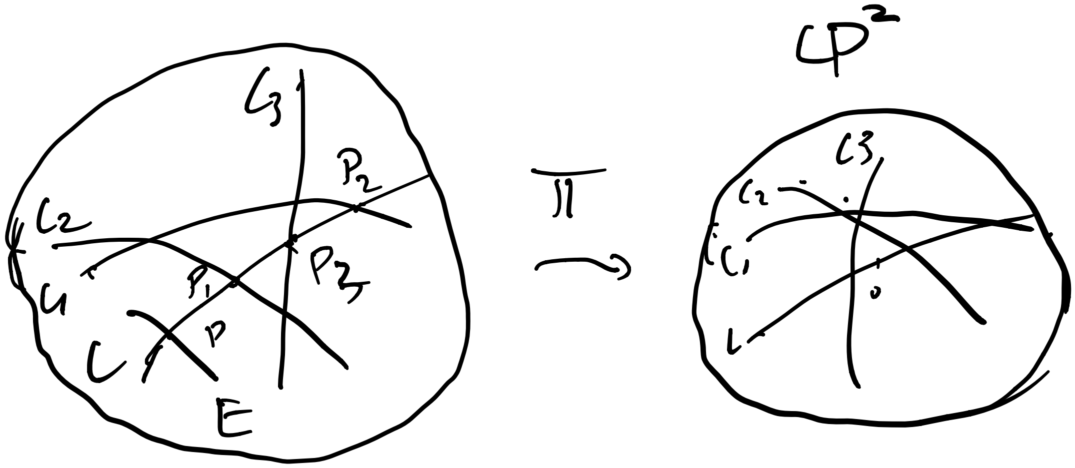
$\Xi := (N_\sigma \| K_X + S + B \|_S) \wedge (B|_S)$

$\wedge$ means take min component by component.

Let $u \in H^0(S, m(K_S + B))$ with

$Z(u) \geq m \Xi$,

Ex, $X = Bl_0\, \mathbb{CP}^2$



choose S to be exceptional

$\Delta := C + \frac{2}{3}(E + C_1 + C_2 + C_3)$, $S := C$

(X, Δ) $\quad K_X + \Delta|_S = K_S + \frac{2}{3}(P + P_1 + P_2 + P_3)$

$\Omega = \frac{2}{3}(P + P_1 + P_2 + P_3)$, $S \simeq \mathbb{CP}^1$

$3(K_S + \Omega) = \mathcal{O}(2)$, which is base point free

However, $|K_X + \Delta|_S$ has a component $\frac{2}{3}P$

Hence, not every $H^0(S, 3l(K_S + \Omega)) \uparrow H^0(X, 3l(K_X + \Delta))$

In this example $\Xi = \frac{2}{3}P$

Essentially, we have

$|3l(K_S + \Omega)| + 2lP = |3l(K_X + \Delta)|_S$

Main technical lemma: (1 step version)

$(X, S+B)$ plt, $K_X+S+B \sim_{\mathbb{Q}} \sum \nu^j w_j$

$B = \sum b^j r_j$.

1) $S = W_1$, $\sum_{j\geq 2} W_j \subseteq B$.

2) $u \in H^0(X, m(K_X+S+B))$

3) Fix smooth metric h on K_X+S+B,

and there exists $\{T_k\}_{k\geq 1} \subset L^1(X)$ s.t

$\Theta_h + i\partial\bar\partial T_k \geq -\frac{\omega}{km}$

$T_k|_S \geq C(m) + \log|u|^{2/m}$

Then there exists $(T_k^{(1)})_{m\geq 1} \in L^1(X)$. s.t

① $\sup_X T_k^{(1)} = 0$

$\Theta_h + i\partial\bar\partial T_k^{(1)} \geq -\frac{\omega}{km}$

② $T_k^{(1)}|_S$ is well defined, and

$T_k^{(1)}|_S \geq \frac{\delta_1}{1+\delta_1} \sup_S T_k - \boxed{C} + \frac{1}{m} \log|u|_h^2$

Where C, δ_1 independ of k.

RK: In this one1step improvement, the main gain is

$$\boxed{\begin{aligned}&\text{From } T_k \geq -C(k) + \log|u|^{2/m}\\ &\text{To } T_k^{(1)} \geq \frac{\delta_1}{1+\delta_1}\sup_S T_k - C + \frac{1}{m}\log|u|^{2/m}\end{aligned}}$$

In particular, $\sup_S T_k^{(1)} \geq \frac{\delta_1}{1+\delta_1}\sup_S T_k - C$

If we use $T_k^{(1)}$ as our new T, we get $T_k^{(2)}$ satisfying

$$T_k^{(2)} \geq \frac{\delta_1}{1+\delta_1}\sup_S T_k^{(1)} - C + \frac{1}{m}\log|u|^{2/m}$$

$$\geq \left(\frac{\delta_1}{1+\delta_1}\right)^2 \sup T_k - \frac{\delta_1}{1+\delta_1}C - C + \log|u|^{2/m}.$$

Note $\left(\frac{\delta_1}{1+\delta_1}\right)^n \to 0$ and $\sum_{i=0}^{+\infty}\left(\frac{\delta_1}{1+\delta_1}\right)^i \leq C$

then no matter what's $\sup_S T_k$

we have $T_k^{(\infty)} \geq -C + \log|u|^{2/m}$.

Since we normalized $\sup T_k^\infty < 0$.
T_k^∞ is uniform quasi-PSH. we can extract limit.

A rough idea I think is following.

You already have $T_k|_S \geq c(k) + \log|u|^{2/m}$.

Look at

$$T_k^{(1)} \geq \frac{\delta_1}{1+\delta_1} \sup_S T_k - c + \frac{1}{m} \log|u|^{2/m}.$$

$$T_k^{(1)} = \tilde{T}_k - \sup_X \tilde{T}_k$$

$$\hat{T}_k = \log \sum_i |u_i| \quad \text{probably } \frac{mk+A}{mk}$$

if $\hat{T}_k$ is the extension of

$u^k \otimes s_i$.

then $\hat{T}_k|_S \sim \log|u|^{2/m}$.

Lemma 5.5. ([Ti]) *Let M be a compact complex manifold and α be a real closed $(1,1)$-form on M. We consider the family of normalized potentials*

$$\mathcal{P} := \{ f \in L^1(X) : \sup_M f = 0 \text{ and } \alpha + \sqrt{-1}\partial\bar{\partial} f \geqslant 0 \}.$$

Then there exist constants $\gamma_H > 0$ and $C_H > 0$ such that

$$\int_M e^{-\gamma_H f}\, dV \leqslant C_H \tag{58}$$

for any $f \in \mathcal{P}$. In addition, the numbers γ_H and C_H are uniform with respect to α.

proof of $\exists\ T_k^{(1)}$ w/

1) $\sup_X T_k^{(1)} = 0$

2) $\Theta_h + i\partial\bar\partial T_k^{(1)} \geq -\frac{1}{mk}\omega$

3) $T_k^{(1)} \geq \frac{\delta_1}{1+\delta_1} \sup_X T_k - C + \log|M|^{2/m}$

where C independent of k.

(δ_1 to be determined).

Let's strat with Observation.

if $\hat{L}_i \in mk(K_X+S+B)+A$

$\hat{L}_i|_S = u^k \otimes S_i \qquad S_i \in A$.

S_i globally generated on S.

Then $\varphi_{min} := \frac{1}{mk} \log\sum_i |\hat{u}_i|_h^2 \sim \log|u|^{2/m}$.

if $\varphi_{min} \leq -\frac{\delta_1}{1+\delta_1} \sup_S \varphi_k$, $he^{-\varphi_k} = T_k$

Then $\varphi_{min} - \sup\varphi_{min} \geq \frac{\delta_1}{1+\delta_1} \sup_S \varphi_k - C + \log|u|^{2/m}$.

Hence we are going to prove.

$$\frac{1}{mk}\log\sum|\hat{u}_i|_h^2 < \frac{-\delta_1}{1+\delta_1}\sup_S\varphi_R$$

We skip lower index i.

By the property of uniform quasi-psh.

we only need

$$\frac{1+\delta_1}{mk}\int\log|\hat{u}_i|_h^2 < -\delta_1\sup_S\varphi_R$$

only need

$$e^{\frac{1+\delta_1}{mk}\int\log|\hat{u}|_h^2} \leq \boxed{\int|\hat{u}|^{\frac{2(1+\delta_1)}{mk}} \leq e^{-\delta_1\sup_S\varphi_R}.}$$

Now let's define a $L^{2(1+\delta)}$ norm for section of.

$$mk(K_X+S+B)+A$$

We will prove $\int|\hat{u}|^2e^{-\Phi}\leq e^{-\delta_1\sup_S\varphi_R}$

for some special $\hat{u}$; where $e^{-\Phi}\geq C$.

Let $\mathcal{E} := \{ \hat{u} \in mk(K_X+S+B)+A$

$\hat{u}|_S = u^k \otimes s$

$\int_X |\hat{u}|^{2\frac{1+\delta}{km}} e^{-\delta\varphi_k - \varphi_B - \varphi_S^\delta - \frac{1}{mk}\varphi_A} < +\infty \}$

$$\|\hat{u}\|_{2(1+\delta)} := \int_X |\hat{u}|^{2\frac{1+\delta}{km}} e^{-\delta\varphi_k - \varphi_B - \varphi_S^\delta - \frac{1+\delta}{mk}\varphi_A}.$$

Step 1: $\mathcal{E} \neq \phi$ (using $\varphi_k|_S \geq \log|u|^{2/m}$, $(S, B|_S)$ is klt)

Step 2: In $\mathcal{E}$, $\exists\ \hat{U}_{min}$ minimizing $L^{2(1+\delta)}$ norm)

(compactness),

Step 3: we prove

$$\int |\hat{U}|_{min}^{2\frac{1+\delta}{km}} e^{-\delta\varphi_R - \varphi_B - \varphi_S^\delta - \frac{1}{mk}\varphi_A} \leq C(-\delta) \sup_S \varphi_k$$

Note $\frac{1+\delta}{km}\hat{u}$ is section of $(K_X+S+B+\frac{A}{km})(1+\delta)$

$$= K_X+S+\delta\underbrace{(K_X+S+B)}_{\delta\varphi_k + \frac{1+\delta}{km}\varphi_A + \varphi_B} + \frac{1+\delta}{km}A + B$$

proof of 3:

$$mK(S+K_X+B)+A \qquad (mK-1-\delta)\varphi_{min}+\delta\varphi_R$$

$$= K_X+S+(mK-1-\delta)\left(S+K_X+B+\frac{A}{mK}\right)+\delta(K_X+S+B)$$

$$+\underbrace{B}_{\varphi_B}+\underbrace{\left(1-\frac{mK-1-\delta}{mK}\right)A}_{\varphi_A}.$$

By new O-T

$\exists\, V \in Km(K_X+S+B)+A,$

$V|_S = u^k \otimes s$

$$\int \frac{|V|^2 e^{-\varphi_S^\delta-\varphi_B-\frac{1+\delta}{mk}\varphi_A-\delta\varphi_R}}{|U_{min}|_h^{\frac{2(mk-1-\delta)}{km}}} \leq Q \qquad (*)$$

Q to be computed.

Let $\Phi := e^{-\varphi_S^{\delta} - \varphi_B - \frac{1+\delta}{mk}\varphi_A - \delta\varphi_R}$

Recall

$$\int |U_{min}|^{2\frac{1+\delta}{km}} e^{-\Phi} \leq \int |V|^{2\frac{1+\delta}{km}} e^{-\Phi}$$

Ex: Show that

$$\int |U_{min}|_h^{2\frac{1+\delta}{km}} e^{-\Phi} \leq Q$$

Note if $|U_{min}|_h \leq |V|_h$ pointwise, trivial, apply (*).

Now we chose δ carefully such that $e^{-\Phi}$ has lower bound.

Then $\int |U_{min}|_h^{2(\frac{1+\delta}{km})} \leq Q$

At last we compute $\mathcal{Q}$

$$\mathcal{Q} := \int_S |h^k \otimes s|^{2\frac{1+\delta}{km}} e^{-\delta\varphi_k - \varphi_B - \frac{1+\delta}{km}\varphi_A}$$

$$\approx \int_S |u|^{2\frac{1+\delta}{m}} e^{-\delta\varphi_k - \varphi_B}$$

By Tian's estimate $\exists\, \gamma_X$ depends on X, and C_X such that if $\partial\bar\partial\varphi_k \geq -\omega$.

then $$\int_X e^{-\gamma_X(\varphi_k - \sup\varphi_k)} \leq C_X$$

$$\leq \left[\int \left(|u|^{2\frac{1+\delta}{m}} e^{-\varphi_B}\right)^p\right]^{1/p}$$ p choosen carefully

$$\left[\int e^{-\delta\varphi_k q}\right]^{1/q}$$

$$\lesssim C\left[\int e^{-\delta(\varphi_k - \sup_S \varphi_k)q - q\delta \sup_S \varphi_k}\right]^{1/q}$$

$$\lesssim C\left[e^{-\delta q \sup \varphi_k}\right]^{1/q} C_{\cancel{x}} \quad (\text{if } \delta q \leq \gamma_X)$$

$$\lesssim e^{-\delta \sup \varphi_k} \quad \#.$$

First choose φ. then q, then δ. small.